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ON THE EXTREME SIZES OF GRAPHS WITH A GIVEN PARTIAL ORDER COMPETITION DIMENSION

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Abstract. Most previous research on the partial order competition dimension has focused on computing the dimension of a given graph. In this paper, we take a new approach by considering graphs with a fixed partial order competition dimension and investigating the extremal numbers of edges and vertices that such graphs can have.

1. Introduction

The notion of the partial order competition dimension was introduced by Choi *et al.* in [1] during their study of the structure of competition graphs derived from d-partial orders. The *competition graph* of a digraph D, which is denoted by $C(D)$, is defined to be a graph having the same vertex set as D and having an edge uv if and only if u and v have a common out-neighbor in D. For an integer $d \geq 0$, a *d-partial order* is a digraph $D = (V, A)$ such that $V \subset \mathbb{R}^d$ and $(u, v) \in A$ if and only if u is less than v componentwise. The authors in [1] introduced the notion of partial order competition dimension as follows:

Definition 1.1. Let G be a graph. If we add some additional isolated vertices to G, it can become the competition graph of a d-partial order for some integer $d \geq 0$. The smallest such d is called the *partial order competition dimension* (poc dimension for short) of G, denoted by $\dim_{\text{poc}}(G)$

Most previous research has focused on computing the poc dimension of a given graph. See [1], [2], [3], and [4] for some examples. In this paper, we take a new approach by considering graphs with a fixed partial order

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competition dimension and investigating the extremal numbers of edges and vertices they can have.

2. Main theorem

For a nonnegative integer d, let $\mathcal{G}_d$ be the family of graphs G with $\dim_{\rm poc}(G) = d$. We introduce some parameters as follows:

$$\begin{aligned} \underline{v}_{\rm poc}(d) &= \min\{|V(G)| : G \in \mathcal{G}_d\}, \\ \overline{e}_{\rm poc}(d;n) &= \max\{|E(G)| : G \in \mathcal{G}_d,\ |V(G)| = n\}, \\ \underline{e}_{\rm poc}(d;n) &= \min\{|E(G)| : G \in \mathcal{G}_d,\ |V(G)| = n\}. \end{aligned}$$

These parameters represent the extremal numbers of vertices and edges for graphs in $\mathcal{G}_d$. Note that the parameter $\overline{v}_{\rm poc}(d) = \max\{|V(G)| : G \in \mathcal{G}_d\}$ is undefined, as graphs in $\mathcal{G}_d$ can have an arbitrary number of vertices by adding isolated vertices at will. In fact, [1] showed that a graph G has $\dim_{\rm poc}(G) = 2$ if and only if it is an interval graph that is neither K_s nor $K_t \cup K_1$ for any positive integers s and t. Thus, the value $\underline{v}_{\rm poc}(2)$ is attained by the edgeless graph I_3 on three vertices. Hence, $\underline{v}_{\rm poc}(2) = |V(I_3)| = 3$ and $\underline{e}_{\rm poc}(2;3) = |E(I_3)| = 0$.

The goal of this paper is to investigate $\underline{e}_{\rm poc}(d;n)$ and $\overline{e}_{\rm poc}(d;n)$. We compute them for the first few values of d as follows:

THEOREM 2.1. *For $d \in \{1, 2, 3\}$, $\underline{e}_{\rm poc}(d;n)$ and $\overline{e}_{\rm poc}(d;n)$ satisfy the following:*

(1) *When $n \geq 2$, we have $\underline{e}_{\rm poc}(1;n) = \binom{n-1}{2}$ and $\overline{e}_{\rm poc}(1;n) = \binom{n}{2}$.*
(2) *When $n \geq 3$, we have $\underline{e}_{\rm poc}(2;n) = 0$ and $\overline{e}_{\rm poc}(2;n) = \binom{n}{2} - 1$.*
(3) *When $n \geq 4$, we have $\underline{e}_{\rm poc}(3;n) = 4$ and $\overline{e}_{\rm poc}(3;n) = \binom{n}{2} - 2$.*

Proof. **Case 1**. According to the results of Choi *et al.* ([1]), a graph G has $\dim_{\rm poc}(G) = 1$ if and only if G is either K_{t+1} or $K_t \cup K_1$ for some positive integer t. Therefore, for an integer $n \geq 2$, the graphs $G \in \mathcal{G}_1$ with $|V(G)| = n$ must be equal to either K_n or $K_{n-1} \cup K_1$. Thus, $\underline{e}_{\rm poc}(1;n) = |E(K_{n-1} \cup K_1)| = \binom{n-1}{2}$ and $\overline{e}_{\rm poc}(1;n) = |E(K_n)| = \binom{n}{2}$.

Case 2. As we mentioned earlier, a graph has $\dim_{\rm poc}(G) = 2$ if and only if G is an interval graph that is neither K_s nor $K_t \cup K_1$ for any positive integers s and t. Therefore, $\underline{e}_{\rm poc}(2;n)$ and $\overline{e}_{\rm poc}(2;n)$ are attained by the edgeless graph I_n and the complete graph K_n minus one edge, respectively. Hence $\underline{e}_{\rm poc}(2;n) = 0$ and $\overline{e}_{\rm poc}(2;n) = \binom{n}{2} - 1$.

Case 3. To compute $\overline{e}_{\rm poc}(3;n)$, we shall try to delete edges from K_n as few as possible to obtain a graph G with $\dim_{\rm poc}(G) = 3$. First we remove an edge $e_1 = \{a, b\}$ from K_n. This does not yield the desired graph, as shown in Case 2. Then we delete another edge $e_2 = \{c, d\}$. Since $n \geq 4$, we can select e_2 so that it is not adjacent to e_1. Then the four vertices a, b, c, d form an induce cycle of length 4 in $K_n - e_1 - e_2$, which implies that $K_n - e_1 - e_2$ is not an interval graph. Therefore $\dim_{\text{poc}}(K_n - e_1 - e_2) \geq 3$. Moreover, $K_n - e_1 - e_2$ can be viewed as a graph obtained from $C_4 = abcda$ by sequentially adding universal vertices, each of which is adjacent to all other vertices. Since $\dim_{\text{poc}}(C_4) = 3$ and adding universal vertices does not increase the poc dimension, we have $\dim_{\text{poc}}(K_n - e_1 - e_2) = 3$. Hence $\overline{e}_{\text{poc}}(3; n) = |E(K_n - e_1 - e_2)| = \binom{n}{2} - 2$. We observe that C_4 is the smallest graph in $\mathcal{G}_3$ and that adding isolated vertices to C_4 does not change the poc dimension. Therefore, $\underline{e}_{\text{poc}}(3; n) = |E(C_4 \cup I_{n-4})| = 4$. □

We note that the condition $n \geq 4$ is necessary for $\underline{e}_{\text{poc}}(3; n)$ and $\overline{e}_{\text{poc}}(3; n)$ because no graph G with $\dim_{\text{poc}}(G) = 3$ can have fewer than four vertices. In fact, the process outlined in the proof of Theorem 2.1 can be continued to compute $\underline{e}_{\text{poc}}(4; n)$, $\overline{e}_{\text{poc}}(4; n)$, $\underline{e}_{\text{poc}}(5; n)$, $\overline{e}_{\text{poc}}(5; n)$, and beyond. Then we might expect all the values of $\underline{e}_{\text{poc}}(d; n)$ to take the form $\binom{n}{2} - f(d)$ for some function $f : \mathbb{N} \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers. However, rather than repeating the process of computing $\underline{e}_{\text{poc}}(d; n)$ for each individual $d \geq 4$, we distill the key idea, which is summarized as follows:

THEOREM 2.2. *Let d be a nonnegative integer. For each $n \geq \underline{v}_{\text{poc}}(d)$,*

$$\overline{e}_{\text{poc}}(d; n) \geq \binom{n}{2} - |E(\overline{G})|$$

where G is a graph that attains $\underline{e}_{\text{poc}}(d; \underline{v}_{\text{poc}}(d))$, and $\overline{G}$ is the complement of G.

Proof. By the hypothesis, G is a graph such that $\dim_{\text{poc}}(G) = d$, $|V(G)| = \underline{v}_{\text{poc}}(d)$, and $|E(G)| = \underline{e}_{\text{poc}}(d; \underline{v}_{\text{poc}}(d))$. Let H be a graph obtained from G by sequentially attaching $n - \underline{v}_{\text{poc}}(d)$ universal vertices to G. By the construction, H consists of n vertices and satisfies $|E(H)| = \binom{n}{2} - |E(\overline{G})|$. Moreover, $\dim_{\text{poc}}(H) = \dim_{\text{poc}}(G) = d$ because adding universal vertices does not affect the poc dimension. Thus, we have $\overline{e}_{\text{poc}}(d; n) \geq |E(H)| = \binom{n}{2} - |E(\overline{G})|$, which completes the proof. □

REMARK 2.3. To illustrate Theorem 2.2, consider the case where $n = 4$. Among the graphs $G \in \mathcal{G}_4$, the one with the fewest vertices and edges is the complete bipartite graph $K_{3,3}$ (see [3]). Therefore, we have

$\underline{v}_{\text{poc}}(4) = 6$ and $\underline{e}_{\text{poc}}(4;6) = 9$. Hence, for $n \geq 6 = \underline{v}_{\text{poc}}(4)$, we have $\overline{e}_{\text{poc}}(4;n) \geq \binom{n}{2} - |E(\overline{K_{3,3}})| = \binom{n}{2} - 6$. From the proof of Theorem 2.2, we know that it is possible to delete six properly selected edges from K_n to obtain a graph $G \in \mathcal{G}_4$ with $|V(G)| = n$. This naturally leads to the question of whether it is possible to construct a graph $G \in \mathcal{G}_4$ with $|V(G)| = n$ by deleting fewer than six edges from K_n.

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